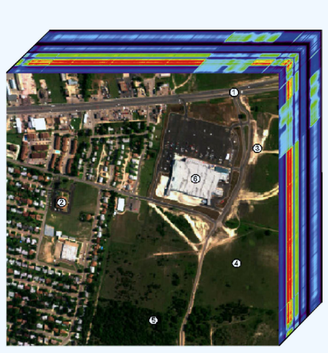
Randomized Successive Projection Algorithm

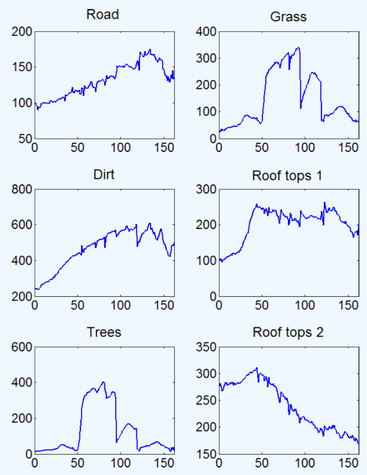
Nonnegative Matrix Factorization (NMF)

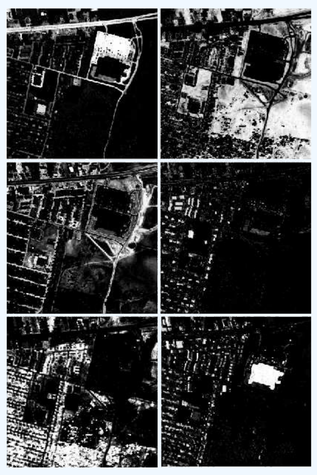
Given an input matrix $X \in \mathbb{R}_+^{m \times n}$ and a factorization rank $r < \min(m, n)$, NMF consists in finding two factors $W \in \mathbb{R}_+^{m \times r}$ and $H \in \mathbb{R}_+^{r \times n}$ such that $X \approx WH$.

Application — Hyperspectral unmixing

$$\underbrace{X(:,j)}_{\text{spectral signature of j-th pixel}} \approx \sum_p \underbrace{W(:,p)}_{\text{spectral signature of p-th material}} \underbrace{H(p,j)}_{\text{abundance of p-th material in j-th pixel}}$$





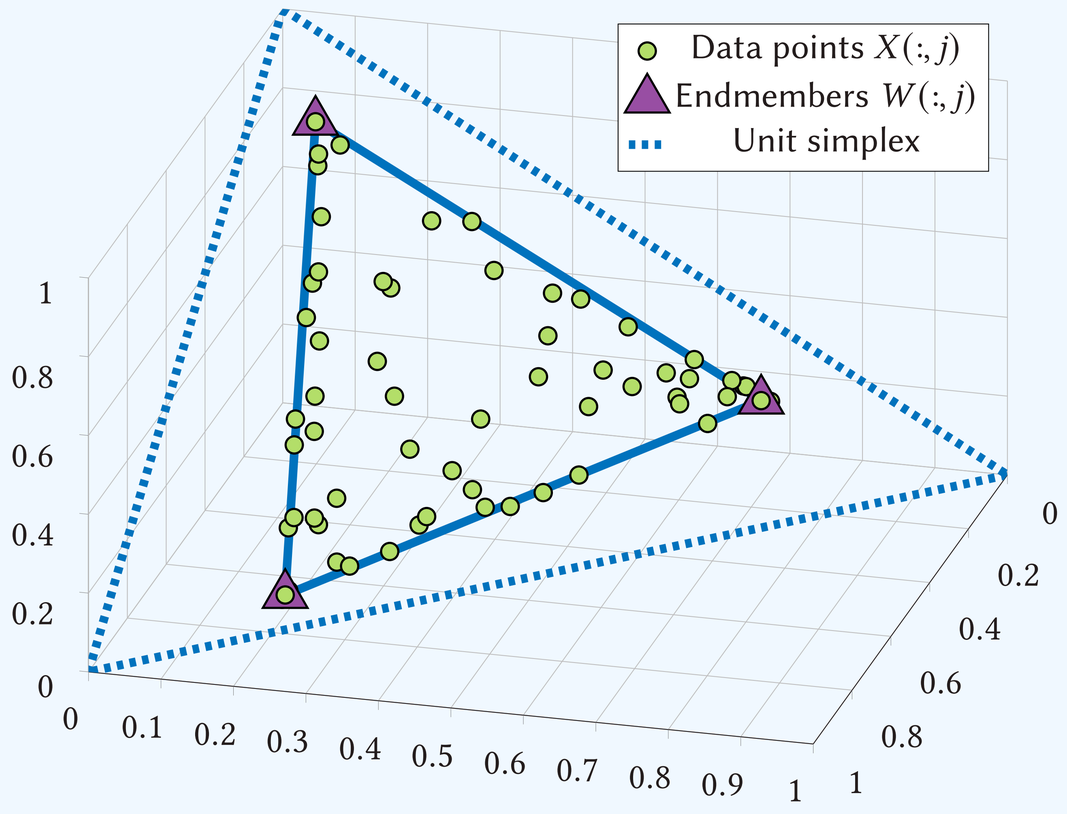


Assumption: Separability

The matrix $X \in \mathbb{R}^{m \times n}$ is r-separable if there exist a subset of columns of X indexed by $\mathcal{J}$ with $|\mathcal{J}| = r$ such that $W = X(:, \mathcal{J})$. Therefore,

$$X \approx X(:, \mathcal{J})H.$$

Separable NMF: find $W = X(:, \mathcal{J})$



Popular separable NMF algorithms

Successive Projection Algorithm (SPA) [Araujo et al, 2001]:

- Deterministic (always output the same result).
- Provably robust to noise.

Vertex Component Analysis (VCA) [Nascimento and Dias, 2005]:

- Random. Can benefit from a multistart strategy.
- Not necessarily robust to noise.

Theorem [N Gillis, WK Ma. Enhancing pure-pixel identification performance via preconditioning, 2015, SIAM J. Imaging Sci.]

Let $\tilde{X} = X + N$, where X is separable, W has full column rank, and N is noise with $\max_j \|N(:,j)\|_2 \leq \epsilon$; and let $Q \in \mathbb{R}^{m \times \nu}$ with $\nu \geq r$. If $Q^\top W$ has full column rank and

$$\epsilon \leq O\left(\frac{\sigma_{\min}(W)}{\sqrt{r}\kappa^3(Q^\top W)}\right),$$

then SPA applied on matrix $Q^\top \tilde{X}$ identifies a set of indices $\mathcal{J}$ corresponding to the columns of W up to the error

$$\max_{1 \leq j \leq r} \min_{k \in \mathcal{J}} \left\| W(:,j) - \tilde{X}(:,k) \right\|_2 \leq O\left(\epsilon\kappa(W)\kappa(Q^\top W)^3\right).$$

Our contribution: Randomized SPA

- Best of both worlds
- Still benefits from provable robustness
- Randomization allows a multi-start strategy

RandSPA

Input: A matrix $X \in \mathbb{R}^{m \times n}$, $r \in \mathbb{N}^*$, ν in $\{1, ..., m\}$.
Output: Index set $\mathcal{J}$ of cardinality r such that $X \approx X(:, \mathcal{J})H$ for some $H \geq 0$.

Let $\mathcal{J} = \emptyset$, $P^\perp = I_m$, $V = [\ ]$.
for $k = 1 : r$ **do**
  Let $Q \in \mathbb{R}^{m \times \nu}$ be a random matrix (with a normal distribution for instance).
  Let $j_k = \operatorname{argmax}_{1 \leq j \leq n} \|Q^\top P^\perp X(:,j)\|_2$ (break ties arbitrarily if needed).
  Let $\mathcal{J} = \mathcal{J} \cup \{j_k\}$.
  Update the projector $P^\perp$ onto the orthogonal complement of $X(:, \mathcal{J})$:

$$v_k = \frac{P^\perp X(:,j_k)}{\|P^\perp X(:,j_k)\|_2},$$
$$V \leftarrow [V\ v_k],$$
$$P^\perp = \left(I_m - VV^T\right).$$

RandSPA coincides with SPA when $Q = I_m$, and with VCA when $\operatorname{rank}(Q) = 1$.

Results on hyperspectral unmixing

Dataset	SPA	Med. RandSPA	Best RandSPA	Med. VCA	Best VCA
Jasper	8.69	8.76	**8.02**	9.47	8.25
Samson	6.49	6.31	**3.97**	6.34	**3.97**
Urban	10.94	9.64	**6.54**	9.61	7.21
Cuprite	2.70	3.53	**2.28**	4.67	2.64

Table: Relative reconstruction error $\|X - WH\|_F / \|X\|_F$ in percent.

(a) SPA (b) RandSPA (c) VCA

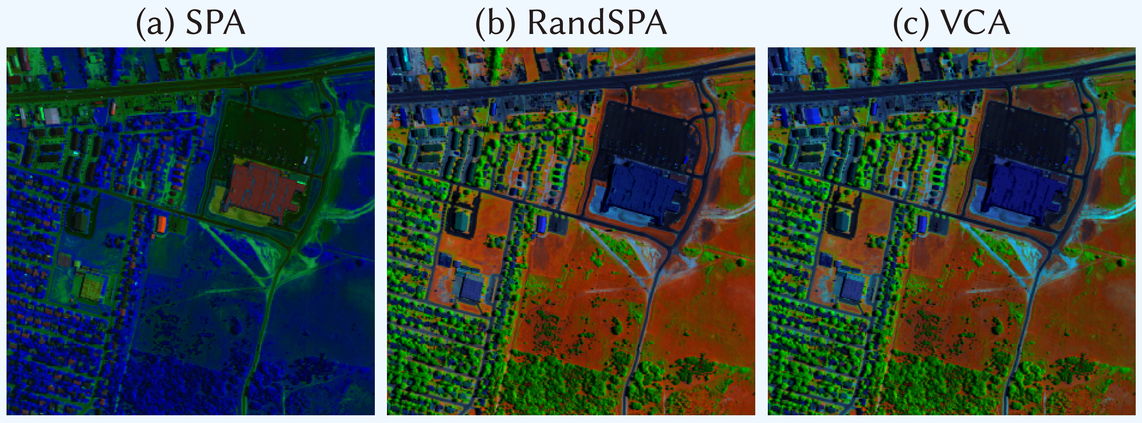


Figure: Abundance maps in false colors from the unmixing of the image Urban.

Robustness to noise

Figure: Average best reconstruction error on several runs, depending on ν, with $\kappa = 1$, on the hyperspectral image Samson with added noise such that $SNR = 20dB$.

Code, refs, and preprint: `https://gitlab.com/nnadisic/randspa`

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